October Math Gems

Problem of the week 17

§1 problems

Problem 1.1. Solve for x

$$\sqrt[4]{1-x^2}+\sqrt[4]{1-x}+\sqrt[4]{1+x}=3$$

Problem 1.2. Find all points (x, y) where the functions $f(x), g(x), h(x)$ have the same value:

$$f(x)=2^{x-5}+3, \qquad g(x)=2x-5, \qquad h(x)=\frac{8}{x}+10$$

Problem 1.3. Solve for x

$$(12x-1)(6x-1)(4x-1)(3x-1)=5$$

Problem 1.4. Solve

$$x^{\left[\frac{3}{4}(\log(x))^2+(\log(x))-\frac{5}{4}\right]}=\sqrt{2}$$

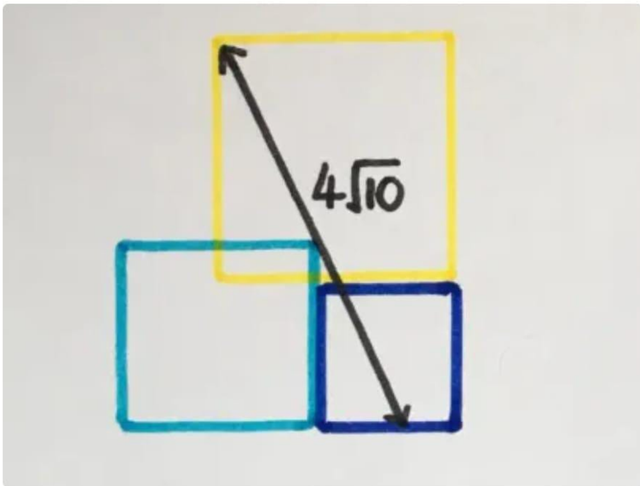
Problem 1.5. Solve for x

$$\frac{2x}{2x^2-5x+3}+\frac{13x}{2x^2+x+3}=6$$

Problem 1.6. If $x^2+y^2=4$, Find the largest value of $3x+4y$.

Problem 1.7. If $ax+(b-3)=(5a-1)x+3b$ has more than one solution, find the value of $100a+4b$.

Problem 1.8. The side lengths of the three squares are consecutive integers. What's the total area?



Problem 1.9. If $6^{-z} = 2^x = 3^y$ then the value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

is ?

Problem 1.10. Find the solution set of the equation

$$3\cos^{-1} x = \sin^{-1}(\sqrt{1-x^2} \times (4x^2 - 1))$$

Problem 1.11. if p, q are odd positive numbers since

$$(1+3+5+\cdots+p) + (1+3+5+\cdots+q) = (1+3+5+\cdots+19)$$

Find the value of $p+q$.

Problem 1.12. Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Problem 1.13. Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

Problem 1.14. Find $x^6 + \frac{1}{x^6}$ if $x + \frac{1}{x} = 3$.

Problem 1.15. Let $a_1 = 2021$ and for $n \geq 1$ let $a_{n+1} = \sqrt{4 + a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}}$$

where m and n are positive integers. Find $10m + n$.

Problem 1.16. The product

$$\left(\frac{1+1}{1^2+1} + \frac{1}{4}\right)\left(\frac{2+1}{2^2+1} + \frac{1}{4}\right)\left(\frac{3+1}{3^2+1} + \frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1} + \frac{1}{4}\right)$$

can be written as $\frac{q}{2^r \cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

Problem 1.17. Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b} + 2\right) + \frac{b}{a}\left(\frac{b}{a} + 2\right) = 2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{n}$$

Problem 1.18. Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Problem 1.19. The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Problem 1.20. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.